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TOPOLOGIZED OBJECTS IN CATEGORIES AND THE SULLIVAN PROFINITE COMPLETION

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The object of this paper is to provide answers to some of the questions raised by J.F. Adams at the Tokyo Topology Conference [2] in connection with Sullivan's profinite completion functor. This functor is defined on the homotopy category $\mathscr{C}$, but takes values in the category $\mathscr{C}^{\mathrm{T}}$ of topologized objects in the homotopy category [1,5,6]. Adams asks the following questions [2]:

(1) Can we set up Sullivan's profinite completion as a functor from $\mathscr{C}^{\mathrm{T}}$ to $\mathscr{C}^{\mathrm{T}}$?

He asserts that it would be sufficient to prove an affirmative answer to:

(2) Does the category $\mathscr{C}^{\mathrm{T}}$ admit weak pull-backs?

(3) Assuming an affirmative answer to (1), can we set up the functor so as to be idempotent, that is, so that the iterated profinite completion is equivalent to the single one?

Affirmative answers to questions (1) and (2) are proved below. Question (3) remains open.

It is a pleasure to express my gratitude to Alex Heller for his suggestions concerning rather radical changes in the emphasis and in the order of ideas in the original draft of this paper; in fact, the present revised version of my manuscript follows closely a rather complete outline prepared by him of a revision of the body of the paper. In particular, the presentation is considerably more concise and makes full use of categorical language.

I would also like to thank Frank Adams for a very kind letter regarding this paper.

1. Categorical preliminaries

We denote by $G\colon \mathrm{Top}\to \mathrm{Sets}$ the forgetful functor. For any category $\mathscr{C}$ and object X of $\mathscr{C}$ a *topologized object over* X is a factorization

$$\begin{tikzcd} \mathscr{C}^{\mathrm{op}} \arrow[r, "X^*"] \arrow[dr, "\mathscr{C}(-,X)"'] & \mathrm{Top} \arrow[d, "G"] \\ & \mathrm{Sets}. \end{tikzcd}$$

These are the objects of the category $\mathscr{C}^{\mathrm{T}}$ of topologized objects of $\mathscr{C}$, morphisms from X^* to Y^* being those $f: X \to Y$ in $\mathscr{C}$ which induce natural transformations $X^* \to Y^*$, that is to say, for which $\mathscr{C}(Z,f)$ is continuous with respect to the topologies of X^*Z, Y^*Z, for all Z in $\mathscr{C}$. We say that X^* is discrete, indiscrete, compact, Hausdorff, etc. if each X^*Z has the property in question. These concepts are introduced in [1] and [2].

The forgetful functor $F: \mathscr{C}^{\mathrm{T}} \to \mathscr{C}$, taking X^* to X, has left and right adjoints D, D' assigning to X the discrete and indiscrete objects over it.

If $\{\phi_\alpha: Y \to GX_\alpha\}$ is a family of maps of sets (indexed by a possibly proper class) the *induced topology* on Y is the coarsest topology for which all ϕ_α are continuous.

Lemma 1.1. *If $X: \mathscr{D} \to \mathrm{Top}$ (where $\mathscr{D}$ need not be small) and GX has a (weak) limit $\{p_d: Y \to GX_d\}$ then the topology induced on Y makes it a (weak) limit of X.*

Proposition 1.2. *The forgetful functor $F: \mathscr{C}^{\mathrm{T}} \to \mathscr{C}$ creates (weak) limits.*

That is to say, if $X: \mathscr{D} \to \mathscr{C}^{\mathrm{T}}$ (where $\mathscr{D}$ need not be small) and FX has a (weak) limit then X itself has a (weak) limit. It is of course gotten by giving to each $\mathscr{C}(Z, \lim FX)$ the induced topology.

Corollary 1.3. *If $\mathscr{C}$ has small (weak) (finite) limits then so has $\mathscr{C}^{\mathrm{T}}$.*

For any subcategory $\mathscr{C}_0$ of $\mathscr{C}$ and any object X in $\mathscr{C}$, $(X \downarrow \mathscr{C}_0)$ will denote [3, p. 46] the category whose objects are maps $f: X \to K$ in $\mathscr{C}$ with K in $\mathscr{C}_0$, and whose morphisms from f to $f': X \to K'$ are those maps $h: K \to K'$ in $\mathscr{C}_0$ for which $hf = f'$.

2. Sullivan's profinite completion

From now on we take $\mathscr{C}$ to be the homotopy category of pointed connected CW-complexes and let $\mathscr{C}_0$ be the full subcategory of those which have finite homotopy groups. We further write $\mathscr{C}_0^{\mathrm{T}} = F^{-1}(\mathscr{C}_0)$. While $\mathscr{C}_0$ is equivalent to a small category, I do not know – it seems unlikely – that this is the case for $\mathscr{C}_0^{\mathrm{T}}$.

Proposition 2.1. *$\mathscr{C}^{\mathrm{T}}$ has products and weak pullbacks preserved by F; $\mathscr{C}_0^{\mathrm{T}}$ has finite products and weak pullbacks preserved by the inclusion $\mathscr{C}_0^{\mathrm{T}} \subset \mathscr{C}^{\mathrm{T}}$.*

This is an immediate consequence of Proposition 1.2 and well-known properties of $\mathscr{C}$, $\mathscr{C}_0$.

We may briefly describe Sullivan's profinite completion in the following way (cf. [1, 5, 6]):

(i) There is a unique functor $R : \mathscr{C}_0 \to \mathscr{C}^{\mathrm{T}}$ such that, for each K in $\mathscr{C}_0$, $FRK = K$ and RK is compact Hausdorff. This functor is full and faithful.

(ii) For any X in $\mathscr{C}$ let $\Phi_X : (X \downarrow \mathscr{C}_0) \to \mathscr{C}$ be the forgetful functor taking $X \to K$ into K.

Then, for any Z in $\mathscr{C}$,

$$\lim \mathscr{C}(Z, \Phi_X -) = G \lim((R\Phi_X -)Z)$$

exists, because $(X \downarrow \mathscr{C}_0)$ is equivalent to a small category. Moreover, if we vary Z in $\mathscr{C}$, we get a contravariant functor from $\mathscr{C}$ to Sets which is half-exact. Thus by Brown's representability theorem $\lim \Phi_X = \mathrm{Su}X$, the (unenriched) *profinite completion,* exists. By Proposition 1.2, in fact, $\mathrm{Su}^{\#}X = \lim R\Phi_X$, the *enriched profinite completion,* also exists, since $FR\Phi_X = \Phi_X$. It is clear from their definitions that these are functors $\mathrm{Su}^{\#} : \mathscr{C} \to \mathscr{C}^{\mathrm{T}}$, $F\mathrm{Su}^{\#} = \mathrm{Su} : \mathscr{C} \to \mathscr{C}$.

The profinite completion comes equipped with a natural transformation $\eta : 1_{\mathscr{C}} \to \mathrm{Su}$, where, for each X in $\mathscr{C}$, $\eta_X : X \to \lim \Phi_X$ is defined by the maps $X \to K$ which are the objects of $(X \downarrow \mathscr{C}_0)$.

3. Two Sullivan completions in $\mathscr{C}^{\mathrm{T}}$

We may copy in $\mathscr{C}^{\mathrm{T}}$ Sullivan's construction in virtue of the following results.

Lemma 3.1. *For any X in $\mathscr{C}^{\mathrm{T}}$ the functor $F_0^X : (X \downarrow \mathscr{C}_0^{\mathrm{T}}) \to (FX \downarrow \mathscr{C}_0)$, taking $X \to K$ to $FX \to FK$, is initial.*

Observe first that it is surjective on objects, since $(FX \to K) = F_0^X(X \to D'K)$. It follows from Proposition 2.1 that $(X \downarrow \mathscr{C}_0^{\mathrm{T}})$ is cofiltered; the conclusion of the lemma is immediate.

Denote by $\Psi_X : (X \downarrow \mathscr{C}_0^{\mathrm{T}}) \to \mathscr{C}^{\mathrm{T}}$ the forgetful functor.

Theorem 3.2. *For any X in $\mathscr{C}^{\mathrm{T}}$, $\lim \Psi_X$ exists.*

For, $F\Psi_X = \Phi_{FX} F_0^X$. But $\lim \Phi_{FX} = \mathrm{Su}FX$ exists. Since F_0^X is initial this is also a limit of $F\Psi_X$ [4, p. 70]. The conclusion follows from Proposition 1.2.

We write $\mathrm{Su}^{\mathrm{T}}X = \lim \Psi_X$; then $\mathrm{Su}^{\mathrm{T}} : \mathscr{C}^{\mathrm{T}} \to \mathscr{C}^{\mathrm{T}}$ is a generalized profinite completion. Of course, $F\mathrm{Su}^{\mathrm{T}} = \mathrm{Su}F$. But Su^{T} and $\mathrm{Su}^{\#}F$ are not a priori comparable.

Corollary 3.3. *For any X in $\mathscr{C}^{\mathrm{T}}$, $\lim RF\Psi_X$ exists.*

For, $FRF\Psi_X = F\Psi_X$. But $\lim F\Psi_X$ exists by the above. The conclusion follows from Proposition 1.2.

We write $\mathrm{Su}^{\mathrm{T}\#}X = \lim RF\Psi_X$; then $\mathrm{Su}^{\mathrm{T}\#} : \mathscr{C}^{\mathrm{T}} \to \mathscr{C}^{\mathrm{T}}$ is also a generalized profinite completion. Of course, $F\mathrm{Su}^{\mathrm{T}\#} = F\mathrm{Su}^{\mathrm{T}}$.

Since F_0^X is initial and $RF\Psi_X = R\Phi_{FX}F_0^X$, we have, for any X in $\mathscr{C}^{\mathrm{T}}$, $\lim RF\Psi_X = \lim R\Phi_{FX}$ [4, p. 70], so that $\mathrm{Su}^{\mathrm{T}\#} = \mathrm{Su}^{\#}F$. This also implies that for every functor $L: \mathscr{C} \to \mathscr{C}^{\mathrm{T}}$, which is right-inverse to F, such as, for example, D or D', we have $\mathrm{Su}^{\#} = \mathrm{Su}^{\mathrm{T}\#}L$.

The generalized profinite completion Su^{T} comes equipped with a natural transformation $\theta: 1_{\mathscr{C}^{\mathrm{T}}} \to \mathrm{Su}^{\mathrm{T}}$, where, for each X in $\mathscr{C}^{\mathrm{T}}$, $\theta_X: X \to \lim \Psi_X$ is defined by the maps $X \to K$ which are the objects of $(X \downarrow \mathscr{C}_0^{\mathrm{T}})$. However, as far as $\mathrm{Su}^{\mathrm{T}\#}$ is concerned, one can only say that the maps $FX \to FK$ define a natural transformation $F\theta: F \to F\mathrm{Su}^{\mathrm{T}\#}$.

Remark. We can replace $\mathscr{C}_0$ throughout by the full subcategory of $\mathscr{C}$ having as objects those CW-complexes whose homotopy groups are finite P-groups, where P is an arbitrary family of primes. We thus obtain Sullivan's P-profinite completion (cf. [5,6]), and corresponding generalizations to $\mathscr{C}^{\mathrm{T}}$.

References

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